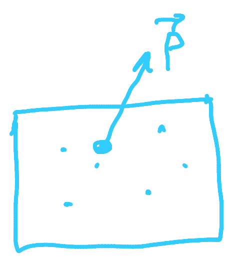
Basics of probability theory

Consider a system which may be in various physical states. For simplicity, assume that these states are <u>discrete</u> and may be labelled 1, 2, 3, ...

Let $X$ be an observable which depends on the state of the system. $X_1, X_2, X_3, \ldots$ – the respective values.

During a long time $T$, the system passes through all of these states. We conduct a series of $N$ measurements, separated by large equal intervals of time




$$w_i = \lim_{N\to\infty} \frac{N_i}{N}$$

– the prob-ty of state $i$

Instead, we may write

$$w_i = \lim_{T\to\infty} \frac{t_i}{T}$$

It is essential that the environment is stationary. Our consideration would not work, for example, for an infinitely expanding gas

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Quite often, one deals with continuous quantities $X$

the probability

qu...
quantities $X$

Then one may define the probability that $X$ is in the interval $(X, X+dX)$ $w(X, X+dX)$

$$w(X, X+dX) = \lim_{T\to\infty} \frac{dt(X, X+dX)}{T}$$

$$dw_x = \rho(X)\, dX$$

Probability density (distribution f-n)

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Often, different definitions are used.
Instead of considering the change of the state of a system in time, one may consider a number of copies of the same system whose states are distributed randomly.
They are called statistical ensemble

$$w_i = \lim_{n\to\infty} \frac{n_i}{n}$$

$n_i$ ← # systems in state i

$n$ ← # systems

Ergodicity: ensembe average = time average

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Basic properties of probability densities

$w_{i+k} = w_i + w_k$ (prove by considering either an ensemble of time intervals)

$_{i+k}$ either an ensemble ...

$$\sum_i w_i = 1$$

The continuous version: $\int \rho(X)\, dX = 1$

The prob-ty that $X$ lies between $a$ and $b$

$$w(a < X < b) = \int_a^b \rho(X)\, dX$$

If we choose another variable $Y$ and there is one-to-one correspondence $X \leftrightarrow Y$ between $X$ and $Y$

$$Y = Y(X)$$

$$w(Y(a) < Y < Y(b)) = \int_{Y(a)}^{Y(b)} \underbrace{\rho(X(Y)) \left|\frac{\partial X}{\partial Y}\right|}_{\rho_Y(Y)} dY$$

$$\rho_x(X) = \rho_Y(Y) \left|\frac{\partial X}{\partial Y}\right|$$

This may be generalised to the case of an arbitrary number of variables

$$\int \rho(X_1, \dots, X_n)\, dX_1 \dots dX_n$$

when changing variables,

$$\rho_x(X_1, \dots, X_n) = \rho_Y(Y_1, \dots, Y_n) \left|\frac{\partial(X_1, \dots, X_n)}{\partial(Y_1, \dots, Y_n)}\right|$$

statistical independence:

Statistical independence:

$\rho(X)dX \, \rho(Y)dY = w(X, X+dX \,;\, Y, Y+dY)$

- $X$ and $Y$ are independent variables

$w_{ik} = w_i w_k$ - the discrete version of it

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Example: the probability density of a molecule in a gas in a closed volume $V$

$$\rho(\vec{r}) = \frac{1}{V}$$

$$w_v = \int \rho(\vec{r}) \, d\vec{r}$$

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The average value of an observable

$\overline{X} \equiv \langle X \rangle = \int X \rho(X) \, dX$

$\overline{X+Y} = \overline{X} + \overline{Y}$

For independent quantities, $\overline{XY} = \overline{X}\,\overline{Y}$